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Hidden $N = (2|2)$ Supersymmetry of the $N = (1|1)$ Supersymmetric Toda Lattice Hierarchy

O. LECHTENFELD † *and A SORIN* ‡

† *Institut für Theoretische Physik, Universität Hannover*
Appelstraße 2, D-30167 Hannover, Germany
E-mail: lechtenf@itp.uni-hannover.de

‡ *Bogoliubov Laboratory of Theoretical Physics, JINR*
141980 Dubna, Moscow Region, Russia
E-mail: sorin@thsun1.jinr.ru

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Abstract

An $N = (2|2)$ superfield formulation of the $N = (1|1)$ supersymmetric Toda lattice hierarchy is proposed, and its five real forms are presented.

1 Introduction

Recently the $N = (1|1)$ supersymmetric generalization [1] of the Darboux transformation [2] was proposed, and an infinite class of bosonic and fermionic solutions of its symmetry equation was constructed in [1, 3]. These solutions generate bosonic and fermionic flows of the $N = (1|1)$ supersymmetric Toda lattice hierarchy in the same way as their bosonic counterparts — the solutions of the symmetry equation of the Darboux transformation [4] — produce the flows of the bosonic Toda lattice hierarchy. Actually, the $N = (1|1)$ Toda lattice hierarchy is $N = (2|2)$ supersymmetric (see Section 3), and henceforth we shall call it the $N = (2|2)$ Toda lattice hierarchy. Naturally, the quest for its $N = (2|2)$ superfield formulation arises.

The present letter addresses this problem. In Section 2 we present a short summary of the main facts concerning the $N = (2|2)$ Toda lattice hierarchy and its bosonic and fermionic flows which are used in what follows. In Section 3 we formulate a *conjecture* concerning the $N = (2|2)$ superfield formulation of the $N = (2|2)$ Toda lattice hierarchy. Our conjecture is partly proven in Section 4 and gains further support in Section 5 by a set of arguments, including explicit calculations of the first three flows. In Section 6 we also present five complex conjugations in $N = (2|2)$ superspace which are admitted by the flows.

2 $N = (2|2)$ Toda lattice hierarchy in $N = (1|1)$ superspace

In this section we briefly review the approach of refs. [1, 3] (for more detail, see [1, 3] and references therein) for constructing an infinite class of bosonic and fermionic flows of the $N = (2|2)$ Toda lattice hierarchy in $N = (1|1)$ superspace.

The starting point is the $N = (1|1)$ supersymmetric generalization of the Darboux transformation [1],

$$u_{j+1} = \frac{1}{v_j}, \qquad v_{j+1} = v_j(D_-D_+ \ln v_j - u_j v_j), \tag{1}$$

where $u_j \equiv u_j(x^+, \theta^+; x^-, \theta^-)$ and $v_j \equiv v_j(x^+, \theta^+; x^-, \theta^-)$ are bosonic $N = (1|1)$ superfields defined on the lattice, $j \in \mathbb{Z}$, and $D_\pm$ are the $N = 1$ supersymmetric fermionic covariant derivatives

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\pm \frac{\partial}{\partial x^\pm}, \qquad D_\pm^2 = \frac{\partial}{\partial x^\pm} \equiv \partial_\pm, \qquad \{D_+, D_-\} = 0. \tag{2}$$

The composite superfield

$$b_j \equiv u_j v_j \tag{3}$$

satisfies the $N = (1|1)$ supersymmetric Toda lattice equation

$$D_-D_+ \ln b_j = b_{j+1} - b_{j-1}. \tag{4}$$

For this reason, the hierarchy of equations invariant under the Darboux transformation (1) is called the $N = (1|1)$ supersymmetric Toda lattice hierarchy.

One of the possible ways of constructing invariant equations is to solve the corresponding symmetry equation. In the case under consideration it reads

$$U_{j+1} = -\frac{1}{v_j^2} V_j, \qquad V_{j+1} = \frac{v_{j+1}}{v_j} V_j + v_j \left(D_-D_+ \left(\frac{1}{v_j} V_j \right) - v_j U_j - u_j V_j \right), \tag{5}$$

where V_j and U_j are bosonic functionals of the superfields v_j and u_j. Any particular solution V_j^p, U_j^p generates an evolution system of equations involving only the superfields v_j and u_j defined at the same lattice point j, with respect to a bosonic evolution time t_p,

$$\frac{\partial}{\partial t_p} v_j = V_j^p, \qquad \frac{\partial}{\partial t_p} u_j = U_j^p. \tag{6}$$

By construction[1], this system is invariant under the discrete transformation (1) and, therefore, belongs to the hierarchy as defined above. In other words, different solutions of the evolution system (6) (which, actually, are given by pairs of superfields $\{v_j, u_j\}$ with different values for j) are related by the discrete Darboux transformation (1). Altogether, invariant evolution systems form a *differential* hierarchy, i.e. a hierarchy of equations involving only superfields at a single lattice point[2]. In contrast, the discrete lattice shift (the Darboux transformation), when added to the differential hierarchy, generates the *discrete*

[1] Let us recall that eq. (5) is just a result of differentiating eq. (1) with respect to the evolution time t_p.

[2] In the case of the one- (two-) dimensional bosonic Toda lattice the differential hierarchy coincides with the nonlinear Schrödinger (Davey–Stewartson) hierarchy [5, 6, 4].

$N = (1|1)$ supersymmetric Toda lattice hierarchy. Thus, the discrete hierarchy appears as a collection of an infinite number of isomorphic differential hierarchies [5].

The symmetry equation (5) represents a complicated nonlinear functional equation, and its general solution is not known. For a more complete understanding of the hierarchy structure and its solutions it seems advantageous to know as many solutions of eq. (5) as possible. Refs. [1, 3] addressed this problem and derived a wide class of bosonic as well as fermionic solutions.

First, the functionals V_j and U_j are consistently represented in terms of a single bosonic functional $\alpha_{0,j}[u_j, v_j]$,

$$V_j = -v_j \alpha_{0,j}, \qquad U_j = u_j \alpha_{0,j-1}, \tag{7}$$

in terms of which the symmetry equation (5) becomes

$$D_- D_+ \alpha_{0,j} = b_{j+1} (\alpha_{0,j+1} - \alpha_{0,j}) + b_j (\alpha_{0,j} - \alpha_{0,j-1}), \tag{8}$$

where the superfield b_j is defined by eq. (3) and constrained by eq. (4).

Second, the following recursive chain of substitutions is introduced:

$$\alpha^{\pm}_{p,j} = \pm D^{-1}_{\mp} \left(b_{j+p+1} \alpha^{\pm}_{p+1,j} + (-1)^p b_j \alpha^{\pm}_{p+1,j-1} \right), \qquad p = 0, 1, 2, \ldots, \tag{9}$$

where $\alpha^{\pm}_{2p,j}$ ($\alpha^{\pm}_{2p+1,j}$) are new bosonic (fermionic) functionals of length dimension

$$[\alpha^{\pm}_{p,j}] = [\alpha^{\pm}_{0,j}] + \frac{p}{2}, \tag{10}$$

and the superscripts $+$ and $-$ mark two different series of solutions to the symmetry equation (8). Equations (9) can be used to express $\alpha^{\pm}_{0,j}$ in terms of $\alpha^{\pm}_{p,j}$ for any chosen p. The following equation for $\alpha^{\pm}_{p,j}$,

$$\begin{aligned} &\pm(-1)^p D_{\pm} \alpha^{\pm}_{p,j} + \alpha^{\pm}_{p,j} D^{-1}_{\mp} (b_{j+p+1} + b_{j+p} - b_{j+1} - b_j) \\ &\quad = D^{-1}_{\mp} \left(b_{j+p+1} \alpha^{\pm}_{p,j+1} + (-1)^p (b_{j+p} - b_{j+1}) \alpha^{\pm}_{p,j} - b_j\, \alpha^{\pm}_{p,j-1} \right), \end{aligned} \tag{11}$$

can easily be proved by induction.

We now describe the solutions of the equations arising in this iterative process. It turns out that, at any given p, the equation (11) possesses a very simple solution for $\alpha^{\pm}_{p,j}$, namely

$$\alpha^{\pm}_{p,j} = (-1)^{pj} \epsilon_p \qquad \Rightarrow \qquad [\alpha^{\pm}_{p,j}] = 0, \tag{12}$$

where ϵ_p is a dimensionless fermionic (bosonic) constant for odd (even) values of p. Therefore, the recursive procedure may be entered at any chosen p with the simple initial value (12), which then generates a very non-trivial solution $\alpha^{(p)\pm}_{0,j}$ for the functional $\alpha^{\pm}_{0,j}$ via (9). The latter, in turn, yields the flows via eqs. (6) and (7).

Let us demonstrate in more detail how bosonic and fermionic flows originate from this background.

For the bosonic functionals $\alpha^{\pm}_{2p,j}$ the recursive procedure may be started at any even step. The corresponding $\alpha^{(2p)\pm}_{0,j}$, being expressed in terms of $\alpha^{\pm}_{2p,j}$ (12) via relations (9), has the following symbolic form [7],

$$\alpha^{(2p)\pm}_{0,j} = \pm \left[\prod_{k=1}^{2p} \left(1 - (-1)^k e^{-\left(k\partial_k + \sum_{n=k+1}^{2p} \partial_n \right)} \right) \right] \left(\prod_{m=1}^{2p} D^{-1}_{\mp} b_{j+m} \right), \tag{13}$$

and generates the p-th bosonic flow of the hierarchy,

$$\frac{\partial}{\partial t_p^{\pm}} v_j = -v_j \alpha_{0,j}^{(2p)\pm}, \qquad \frac{\partial}{\partial t_p^{\pm}} u_j = u_j \alpha_{0,j-1}^{(2p)\pm} \qquad \Rightarrow \qquad [t_p^{\pm}] = -[\alpha_{0,j}^{(2p)\pm}] = p, \tag{14}$$

where we have used eqs. (6), (7), (10), and (12), and the superscripts + and − correspond to the two different series (9) of solutions to the symmetry equation (8). The operator $e^{-l\partial_k}$ ($l \in \mathbb{Z}$) is the discrete lattice shift which acts in eq. (13) according the rule

$$e^{-l\partial_k} \left(\prod_{m=1}^{2p} D_{\mp}^{-1} b_{j+m} \right) := \left(\prod_{m=1}^{k-1} D_{\mp}^{-1} b_{j+m} \right) D_{\mp}^{-1} b_{j+k-l} \left(\prod_{m=k+1}^{2p} D_{\mp}^{-1} b_{j+m} \right), \tag{15}$$

i.e. ∂_k is meant to act only on b_{j+k} in the product. By definition, the lattice shift operator $e^{-l\partial_k}$ commutes with the fermionic covariant derivatives $D_{\pm}$,

$$[e^{-l\partial_k}, D_{\pm}] = 0. \tag{16}$$

Although the solution $\alpha_{0,j}^{(2p)\pm}$ depends on all superfields v_{j+k} and u_{j+k} with $0 \le k \le 2p$, by using eq. (1) it can be expressed completely in terms of the superfields u_j and v_j defined at the single lattice point j. In this way the differential hierarchy of bosonic flows (14) is generated (see the discussion after eq. (6)). For illustration, we present the first two [1]:

$$\frac{\partial}{\partial t_1^{+}} v = v, \qquad \frac{\partial}{\partial t_1^{+}} u = u, \tag{17}$$

$$\begin{aligned} \frac{\partial}{\partial t_2^{+}} v &= +\partial_+^2 v - 2(D_+ v) D_-^{-1} \partial_+ (uv) + 2v D_-^{-1} \left[\partial_+ (v D_+ u) + 2uv D_-^{-1} \partial_+ (uv) \right], \\ \frac{\partial}{\partial t_2^{+}} u &= -\partial_+^2 u - 2(D_+ u) D_-^{-1} \partial_+ (uv) + 2u D_-^{-1} \left[\partial_+ (u D_+ v) - 2uv D_-^{-1} \partial_+ (uv) \right], \end{aligned} \tag{18}$$

where $u \equiv u_j(x^+, \theta^+; x^-, \theta^-)$ and $v \equiv v_j(x^+, \theta^+; x^-, \theta^-)$.

For the fermionic functionals $\alpha_{2p-1,j}^{\pm}$ the recursive procedure may be started at any odd step. It remains to show how fermionic flows are being activated. This goal in mind, let us represent the bosonic time derivative entering eq. (6) in the following form:

$$\frac{\partial}{\partial t_p} = \epsilon_{2p-1} D_p, \tag{19}$$

defining a fermionic time derivative D_p. Then, eq. (6) becomes

$$\begin{aligned} &\epsilon_{2p-1} D_p^{\pm} v_j = -v_j \alpha_{0,j}^{(2p-1)\pm}, \quad \epsilon_{2p-1} D_p^{\pm} u_j = u_j \alpha_{0,j-1}^{(2p-1)\pm} \\ &\qquad \Rightarrow \quad [D_p^{\pm}] = [\alpha_{0,j}^{(2p-1)\pm}] = -p + \tfrac{1}{2}, \end{aligned} \tag{20}$$

where $\alpha_{0,j}^{(2p-1)\pm}$ should be expressed in terms of $\alpha_{2p-1,j}^{\pm}$ (12) via relations (9), and eqs. (7), (10) and (12) have been exploited to arrive at eqs. (20). The superscripts on $D_p^{\pm}$ in eqs. (20) again correspond to the two different series (9) of solutions to the symmetry equation (8). The fermionic constant ϵ_{2p-1} enters linearly both sides of eqs. (20), hence the fermionic flows $D_p^{\pm}$ actually do not depend on ϵ_{2p-1}. In this context we remark that

ϵ_{2p-1} is an artificial parameter which need not be introduced at all. However, without ϵ_{2p-1} it is necessary to consider the quantities t_p, V_j^p, U_j^p, $\alpha^{\pm}_{2p,j}$ ($\alpha^{\pm}_{2p-1,j}$) entering eqs. (6), (9) as fermionic (bosonic) ones from the beginning. Of course, at the end of the analysis one arrives at the same result (20). For illustration, we present the first two fermionic flows from the set (20) [3]:

$$(-)^j D_1^+ v = -D_+ v + 2vD_-^{-1}(uv), \qquad (-)^j D_1^+ u = -D_+ u - 2uD_-^{-1}(uv), \tag{21}$$

$$\begin{aligned}(-)^j D_2^+ v = {} & -D_+\partial_+ v + 2(\partial_+ v)D_-^{-1}(uv) \\ & + (D_+ v)D_-^{-1}D_+(uv) + vD_-^{-1}[u\partial_+ v + (D_+ v)D_+ u], \\ (-)^j D_2^+ u = {} & +D_+\partial_+ u + 2(\partial_+ u)D_-^{-1}(uv) \\ & + (D_+ u)D_-^{-1}D_+(uv) + uD_-^{-1}[v\partial_+ u + (D_+ u)D_+ v].\end{aligned} \tag{22}$$

Let us note that the two differential hierarchies arising for the two different values of $(-1)^j$ ($+1$ or -1) are actually isomorphic. Indeed, one can easily see that they are related by the standard automorphism which changes the sign of all Grassmann numbers. Thus, in distinction the bosonic Toda lattice, where the Darboux transformation does not change the direction of evolution times in the differential hierarchy (6), its supersymmetric counterpart (1) reverses the sign of fermionic times in the differential hierarchy. This supersymmetric peculiarity has no effect on the property that the supersymmetric *discrete* hierarchy is a collection of isomorphic *differential* hierarchies like in the bosonic case[3].

The flows D_k^- and $\frac{\partial}{\partial t_k^-}$ can easily be derived by applying the invariance transformations

$$\partial_{\pm} \longrightarrow \partial_{\mp}, \qquad D_{\pm} \longrightarrow \pm D_{\mp} \tag{23}$$

of the $N = (1|1)$ supersymmetry algebra (2) and eqs. (1), (4) and (8) to the flows D_k^+ (21)–(22) and $\frac{\partial}{\partial t_k^+}$ (17)–(18), respectively, but we do not write them down here.

Using the explicit expressions for the constructed bosonic and fermionic flows, one can calculate their algebra

$$\begin{aligned} & \{D_k^{\pm}, D_l^{\pm}\} = -2\frac{\partial}{\partial t^{\pm}_{k+l-1}}, \\ & \{D_k^+, D_l^-\} = \left[\frac{\partial}{\partial t_k^{\pm}}, \frac{\partial}{\partial t_l^{\pm}}\right] = \left[\frac{\partial}{\partial t_k^{+}}, \frac{\partial}{\partial t_l^{-}}\right] = \left[\frac{\partial}{\partial t_k^{\pm}}, D_l^{\pm}\right] = \left[\frac{\partial}{\partial t_k^{\pm}}, D_l^{\mp}\right] = 0, \end{aligned} \tag{24}$$

which may be realized in the superspace $\{t_k^+, \theta_k^+; t_k^-, \theta_k^-\}$ via

$$D_k^{\pm} = \frac{\partial}{\partial \theta_k^{\pm}} - \sum_{l=1}^{\infty} \theta_l^{\pm} \frac{\partial}{\partial t^{\pm}_{k+l-1}}, \tag{25}$$

where θ_k^+ and θ_k^- are abelian fermionic evolution times.

[3] For the one-dimensional bosonic Toda lattice hierarchy the isomorphism which relates the differential hierarchies is trivial because they are identical copies of the single nonlinear Schrödinger hierarchy [5].

3 $N = (2|2)$ Toda lattice hierarchy in $N = (2|2)$ superspace

In $N = (1|1)$ superspace, the additional supersymmetry of the $N = (2|2)$ Toda lattice hierarchy is not manifest. Yet, besides the two fermionic flows $D_1^{\pm}$ in (21) and (23), there exist two more fermionic flows $Q_1^{\pm}$. These are generated by the two obvious solutions of the symmetry equation (5) which originate from the standard supersymmetric transformations of the superfields,

$$Q_1^{\pm} v = Q_{\pm} v, \qquad Q_1^{\pm} u = Q_{\pm} u, \tag{26}$$

where $Q_{\pm}$ are $N = (1|1)$ supersymmetric generators,

$$\begin{aligned} &Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - \theta^{\pm} \frac{\partial}{\partial x^{\pm}}, \qquad Q_{\pm}^2 = -\partial_{\pm}, \qquad \{Q_+, Q_-\} = 0, \\ &\{Q_+, D_{\pm}\} = 0, \qquad \{Q_-, D_{\pm}\} = 0. \end{aligned} \tag{27}$$

Altogether, the flows $\left\{\frac{\partial}{\partial t_1^{\pm}}, Q_1^{\pm}, D_1^{\pm}\right\}$ form the superalgebra of complex $N = (2|2)$ supersymmetry. It will turn out that one of the real forms of the hierarchy realizes the *real* $N = (2|2)$ supersymmetry algebra on its flows (see the discussion after eq. (56)).

The existence of the hidden $N = (2|2)$ supersymmetry naturally raises the problem of finding a very particular basis (if any), where it is realized locally and linearly. Its solution would correspond to constructing an $N = (2|2)$ superfield formulation of the hierarchy. With this aim in mind, it is instructive to rewrite the equations (1) and (14) to the new superfield basis $\{J_j, \overline{J}_j\}$,

$$\overline{J}_j := -u_j v_j \equiv -b_j, \qquad J_j := u_j v_j + D_- D_+ \ln u_j, \tag{28}$$

which possesses the above-mentioned properties:

$$\begin{aligned} &Q_1^{\pm} J_j = Q_{\pm} J_j, \qquad (-1)^j D_1^{\pm} J_j = +D_{\pm} J_j, \\ &Q_1^{\pm} \overline{J}_j = Q_{\pm} \overline{J}_j, \qquad (-1)^j D_1^{\pm} \overline{J}_j = -D_{\pm} \overline{J}_j, \end{aligned} \tag{29}$$

where eqs. (21), (23) and (28) have been used. The new superfields $J_j \equiv J_j(x^+, \theta^+; x^-, \theta^-)$ and $\overline{J}_j \equiv \overline{J}_j(x^+, \theta^+; x^-, \theta^-)$ are unconstrained bosonic $N = (1|1)$ lattice superfields. They are related by

$$J_{j+1} = \overline{J}_j, \qquad \overline{J}_{j+1} = J_j - D_- D_+ \ln \overline{J}_j \tag{30}$$

and satisfy

$$\frac{\partial}{\partial t_p^{\pm}} \overline{J}_j = \overline{J}_j \left(\alpha_{0,j-1}^{(2p)\pm} - \alpha_{0,j}^{(2p)\pm}\right), \qquad \frac{\partial}{\partial t_p^{\pm}} J_j = J_j \left(\alpha_{0,j-2}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm}\right), \tag{31}$$

with $\alpha_{0,j}^{(2p)\pm}$ now being understood as functionals of J_j and $\overline{J}_j$.

At this point we formulate our *conjecture*. We claim that the sought-for $N = (2|2)$ superspace formulation is achieved simply by elevating the $N = (1|1)$ lattice superfields J_j and $\overline{J}_j$ to chiral resp. antichiral bosonic $N = (2|2)$ lattice superfields $\mathcal{J}_j(x^+, \theta^+, \eta^+; x^-, \theta^-, \eta^-)$ and $\overline{\mathcal{J}}_j(x^+, \theta^+, \eta^+; x^-, \theta^-, \eta^-)$. More concretely, the resulting equations

$$\mathcal{J}_{2(j+1)} = \mathcal{J}_{2j} - \mathcal{D}_- \mathcal{D}_+ \ln \overline{\mathcal{J}}_{2j}, \qquad \overline{\mathcal{J}}_{2(j+1)} = \overline{\mathcal{J}}_{2j} - \overline{\mathcal{D}}^- \overline{\mathcal{D}}^+ \ln \mathcal{J}_{2(j+1)} \tag{32}$$

and

$$\frac{\partial}{\partial t_p^{\pm}} \overline{\mathcal{J}}_j = \overline{\mathcal{J}}_j \left(\alpha_{0,j-1}^{(2p)\pm} - \alpha_{0,j}^{(2p)\pm} \right), \qquad \frac{\partial}{\partial t_p^{\pm}} \mathcal{J}_j = \mathcal{J}_j \left(\alpha_{0,j-2}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm} \right) \tag{33}$$

are conjectured to be consistent with the chirality constraints

$$\overline{\mathcal{D}}^{\pm} \overline{\mathcal{J}}_{2j} = \mathcal{D}_{\pm} \overline{\mathcal{J}}_{2j+1} = 0 \qquad \text{and} \qquad \mathcal{D}_{\pm} \mathcal{J}_{2j} = \overline{\mathcal{D}}^{\pm} \mathcal{J}_{2j+1} = 0. \tag{34}$$

We would like to emphasize that the last statement is no trivial matter because such a procedure in general leads to inconsistent equations except for very special cases one of which is under consideration. In the above, $\mathcal{D}_{\pm}$ and $\overline{\mathcal{D}}^{\pm}$ are $N = (2|2)$ supersymmetric fermionic covariant derivatives,

$$\begin{aligned}
&\mathcal{D}_{\alpha} := \frac{1}{2} \left(\frac{\partial}{\partial \theta^{\alpha}} + i \frac{\partial}{\partial \eta^{\alpha}} + (\theta^{\alpha} + i \eta^{\alpha}) \, \partial_{\alpha} \right), \\
&\overline{\mathcal{D}}^{\alpha} := \frac{1}{2} \left(\frac{\partial}{\partial \theta^{\alpha}} - i \frac{\partial}{\partial \eta^{\alpha}} + (\theta^{\alpha} - i \eta^{\alpha}) \, \partial_{\alpha} \right), \qquad \alpha, \beta = \pm, \\
&D_{\pm} = \mathcal{D}_{\pm} + \overline{\mathcal{D}}^{\pm}, \qquad \left\{ \mathcal{D}_{\alpha}, \overline{\mathcal{D}}^{\beta} \right\} = \delta_{\alpha}{}^{\beta} \partial_{\beta}, \qquad \{ \mathcal{D}_{\alpha} \, , \, \mathcal{D}_{\beta} \} = \left\{ \overline{\mathcal{D}}^{\alpha}, \overline{\mathcal{D}}^{\beta} \right\} = 0,
\end{aligned} \tag{35}$$

and $\eta^{\pm}$ are two additional Grassmanian coordinates. Since the right hand sides of eqs. (33) are solutions of the symmetry equation corresponding to eqs. (32), we must require that the functionals $\alpha_{0,j}^{\pm} - \alpha_{0,j-1}^{\pm}$ entering eqs. (33) possess the following chirality properties:

$$\overline{\mathcal{D}}^{\mp} \left(\alpha_{0,2j}^{(2p)\pm} - \alpha_{0,2j-1}^{(2p)\pm} \right) = 0, \qquad \mathcal{D}_{\mp} \left(\alpha_{0,2j+1}^{(2p)\pm} - \alpha_{0,2j}^{(2p)\pm} \right) = 0, \tag{36}$$

$$\overline{\mathcal{D}}^{\pm} \left(\alpha_{0,2j}^{(2p)\pm} - \alpha_{0,2j-1}^{(2p)\pm} \right) = 0, \qquad \mathcal{D}_{\pm} \left(\alpha_{0,2j+1}^{(2p)\pm} - \alpha_{0,2j}^{(2p)\pm} \right) = 0. \tag{37}$$

These four equations are necessary and sufficient conditions for the consistency of (32) and (33) with the constraints (34). Hence, we should set out to prove them.

4 Proof of half the conjecture

In the following, we present a *proof* that the constraints (36) are in fact satisfied. What concerns the remaining constraints (37), we shall give evidence in their favour in the next section, by confirming them (and (36)) explicitly for the first three flows from the set (33).

First, the equations (32) are obviously consistent with the chirality constraints (34) and represent a manifestly $N = (2|2)$ supersymmetric form of the $N = (2|2)$ supersymmetric Toda lattice equations (see, e.g. refs. [8, 9] and references therein). From the chirality constraints (34) one can derive the following intertwining relations of the fermionic covariant derivatives $\mathcal{D}_{\pm}$ and $\overline{\mathcal{D}}^{\pm}$ with the lattice shift operator e^{∂}:

$$e^{\partial} \overline{\mathcal{D}}^{\mp} = \mathcal{D}_{\mp} \, e^{\partial}, \qquad e^{\partial} \, \mathcal{D}_{\mp} = \overline{\mathcal{D}}^{\mp} e^{\partial}, \tag{38}$$

which are obviously consistent with the commutation relations (16) by way of $D_{\pm} = \mathcal{D}_{\pm} + \overline{\mathcal{D}}^{\pm}$ (35). From these relations one can easily see that fermionic covariant derivatives commute with the shifts by *even* number of lattice points only. Therefore, the chirality

constraints (34) are invariant with respect to shifts by only an *even* number of lattice points, which is the reason why only superfields $\{\mathcal{J}_{2j}, \overline{\mathcal{J}}_{2j}\}$ at *even* lattice points enter the equations (32). In spite of this peculiarity the numbers of independent dynamical degrees of freedom entering the $N=(1|1)$ equations (30) and the $N=(2|2)$ equations (32) are the same, and they are in one-to-one correspondence.

Second, using eqs. (13) and (28), after obvious manipulations the functionals $\alpha_{0,j}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm}$ can identically be represented in the following form:

$$\begin{aligned}
\alpha_{0,j}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm} &= \left(1 - e^{-\sum_{n=1}^{2p}\partial_n}\right)\alpha_{0,j}^{\pm} \\
&= \pm\left(1 - e^{-2\sum_{n=1}^{2p}\partial_n}\right)\left(1 - e^{-2p\partial_{2p}}\right)\left\{\prod_{k=1}^{p-1}\left(1 - e^{-\left(2k\partial_{2k} + \sum_{n=2k+1}^{2p}\partial_n\right)}\right)\right. \\
&\quad\times\left.\left(1 + e^{-\left((2k+1)\partial_{2k+1} + \sum_{n=2(k+1)}^{2p}\partial_n\right)}\right)\right\}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right).
\end{aligned} \tag{39}$$

Then, we find explicitly a product of the expressions inside the first two brackets in the second line of eq. (39) and use the identity

$$\begin{aligned}
&\mathcal{P}_{k-1,k}\, e^{-\left((k-1)\partial_{k-1} + \sum_{n=k}^{2p}\partial_n\right)}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) \\
&\qquad = e^{-\left(k\partial_k + \sum_{n=k+1}^{2p}\partial_n\right)}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right),
\end{aligned} \tag{40}$$

where $\mathcal{P}_{k-1,k}$ is a permutation operator which acts according the rule

$$\begin{aligned}
&\mathcal{P}_{k-1,k}\, e^{l\partial_k} = e^{l\partial_{k-1}}\mathcal{P}_{k-1,k}, \qquad \mathcal{P}_{k-1,k}\, e^{l\partial_{k-1}} = e^{l\partial_k}\mathcal{P}_{k-1,k}, \\
&\mathcal{P}_{k-1,k}\left(\prod_{m=1}^{k-2} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) D_{\mp}^{-1}\overline{\mathcal{J}}_l D_{\mp}^{-1}\overline{\mathcal{J}}_n \left(\prod_{m=k+1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) \\
&= \left(\prod_{m=1}^{k-2} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) D_{\mp}^{-1}\overline{\mathcal{J}}_n D_{\mp}^{-1}\overline{\mathcal{J}}_l \left(\prod_{m=k+1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right).
\end{aligned} \tag{41}$$

Equation (39) now reads

$$\begin{aligned}
\alpha_{0,j}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm} &= \pm\left(1 - e^{-2\sum_{n=1}^{2p}\partial_n}\right)\left(1 - e^{-2p\partial_{2p}}\right) \\
&\times\left\{\prod_{k=1}^{p-1}\left(1 - e^{-2\left(k\partial_{2k} + (k+1)\partial_{2k+1} + \sum_{n=2(k+1)}^{2p}\partial_n\right)} + P_{2k,2k+1}\right)\right\}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right),
\end{aligned} \tag{42}$$

with

$$P_{2k,2k+1} := (\mathcal{P}_{2k,2k+1} - 1)e^{-\left(2k\partial_{2k} + \sum\limits_{n=2k+1}^{2p} \partial_n\right)}. \tag{43}$$

A simple inspection of this formula shows that for the validity of the chirality constraints (36) it suffices that the functionals

$$\begin{aligned}
&\mathcal{F}_j^{(2p)\pm} := \left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right), \\
&\mathcal{F}_j^{(l;2p)\pm} := P_{2l,2l+1}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) = P_{2l,2l+1}\mathcal{F}_j^{(2p)\pm}, \quad 1 \le l \le p-1, \\
&\mathcal{F}_j^{(kl;2p)\pm} := P_{2k,2k+1}P_{2l,2l+1}\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) = P_{2k,2k+1}\mathcal{F}_j^{(l;2p)\pm}, \\
&\qquad\qquad 1 \le k < l \le p-1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\mathcal{F}_j^{(m\ldots kl;2p)\pm} := \left(\prod_{k=1}^{p-1} P_{2k,2k+1}\right)\left(\prod_{m=1}^{2p} D_{\mp}^{-1}\overline{\mathcal{J}}_{j+m}\right) = P_{2m,2m+1}\mathcal{F}_j^{(\ldots kl;2p)\pm}, \\
&\qquad\qquad 1 \le m < \cdots < k < l \le p-1
\end{aligned} \tag{44}$$

appearing in (42) satisfy the same chirality constraints[4], i.e.

$$\overline{\mathcal{D}}^{\mp}\mathcal{F}_{2j}^{(m\ldots l;2p)\pm} = 0, \qquad \mathcal{D}_{\mp}\mathcal{F}_{2j+1}^{(m\ldots l;2p)\pm} = 0. \tag{45}$$

Indeed, $\alpha_{0,j}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm}$ (42) is a linear functional of $\mathcal{F}_j^{m\ldots l(2p)\mp}$. The latter can be shifted with lattice shift operators, but only by an even number of lattice points, which does not change the chirality properties of the functionals $\mathcal{F}_j^{m\ldots l(2p)\pm}$.

In trying to verify that the functionals $\mathcal{F}_j^{m\ldots l(2p)\pm}$ (44) do in fact satisfy the conditions (45), we substitute $D_{\pm} = \mathcal{D}_{\pm} + \overline{\mathcal{D}}^{\pm}$ (35) into eqs. (44) and use the relations (34) and (35) in order to simplify the resulting expressions. Let us discuss the outcome:

The functionals $\mathcal{F}_{2j}^{(2p)\mp}$ and $\mathcal{F}_{2j+1}^{(2p)\mp}$ become

$$\begin{aligned}
\mathcal{F}_{2j}^{(2p)\mp} &= \left(\prod_{m=1}^{p} \overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2m-1}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2m}\right) = \overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+1}\mathcal{F}_{2j+1}^{(2(l-1))\mp}\mathcal{D}_{\mp} \\
&\quad \times\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l+1}\mathcal{F}_{2j+2l+1}^{(2(p-l-1))\mp}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2p}, \\
\mathcal{F}_{2j+1}^{(2p)\mp} &= \left(\prod_{m=1}^{p} \mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2m}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2m+1}\right) = \mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2}\mathcal{F}_{2j+2}^{(2(l-1))\mp}\overline{\mathcal{D}}^{\mp} \\
&\quad \times\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l+1}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l+2}\mathcal{F}_{2j+2l+2}^{(2(p-l-1))\mp}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2p+1}
\end{aligned} \tag{46}$$

[4]Let us remark that this is not a necessary condition because the superfields $\{\mathcal{J}_{2j}, \overline{\mathcal{J}}_{2j}\}$ at different lattice points are not linearly independent due to the $N = (2|2)$ Toda lattice equations (32) which relate them.

and satisfy manifestly the conditions (45), due to $(\overline{\mathcal{D}}^{\mp})^2 = 0 = (\mathcal{D}_{\mp})^2$.

It turns out that the product of the operators $P_{2k,2k+1} \dots P_{2l,2l+1}$ $(1 \leq k < \dots < l \leq p-1)$ does not change the chirality properties of $\mathcal{F}^{(2p)\mp}_{2j}$ or $\mathcal{F}^{(2p)\mp}_{2j+1}$ when applied to these functionals. Hence, all the functionals $\mathcal{F}^{m\dots l(2p)\mp}_{j}$ (44) possess the same chirality properties (45) as $\mathcal{F}^{(2p)\mp}_{2j}$ resp. $\mathcal{F}^{(2p)\mp}_{2j+1}$. In order to illustrate this fact, let us present the functionals $\mathcal{F}^{(l;2p)\mp}_{2j}$ and $\mathcal{F}^{(l;2p)\mp}_{2j+1}$ (44),

$$\begin{aligned}
\mathcal{F}^{(l;2p)\mp}_{2j} &= P_{2l,2l+1}\mathcal{F}^{(2p)\pm}_{2j} = \overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+1}\mathcal{F}^{(2(l-1))\mp}_{2j+1}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\Big(\overline{\mathcal{J}}_{2j+2l}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j} \\
&\quad -\overline{\mathcal{J}}_{2j}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l}\Big)\mathcal{F}^{(2(p-k-1))\mp}_{2j+2l}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2p-1}, \\
\mathcal{F}^{(l;2p)\mp}_{2j+1} &= P_{2l,2l+1}\mathcal{F}^{(2p)\pm}_{2j+1} = \mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2}\mathcal{F}^{(2(l-1))\mp}_{2j+2}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\Big(\overline{\mathcal{J}}_{2j+2l+1}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+1} \\
&\quad -\overline{\mathcal{J}}_{2j+1}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l+1}\Big)\mathcal{F}^{(2(p-l-1))\mp}_{2j+2l+1}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2p}.
\end{aligned} \tag{47}$$

Again, they obviously satisfy the conditions (45).

One important remark is in order. When calculating eqs. (47) we have essentially used the following important identities,

$$\begin{aligned}
&\overline{\mathcal{D}}^{\mp}\left(\overline{\mathcal{J}}_{2j+2l}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j} - \overline{\mathcal{J}}_{2j}\mathcal{D}_{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l}\right)\overline{\mathcal{D}}^{\mp} = 0, \\
&\mathcal{D}_{\mp}\left(\overline{\mathcal{J}}_{2j+2l+1}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+1} - \overline{\mathcal{J}}_{2j+1}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+2l+1}\right)\mathcal{D}_{\mp} = 0,
\end{aligned} \tag{48}$$

which can easily be checked using the chirality constraints (34) and the algebra of the fermionic covariant derivatives $\mathcal{D}_{\mp}$ and $\overline{\mathcal{D}}^{\mp}$ (35). Actually, structures like $\overline{\mathcal{D}}^{\mp}\overline{\mathcal{J}}_{2j+2l}\mathcal{D}_{\mp} \times\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j}$ and $\mathcal{D}_{\mp}\overline{\mathcal{J}}_{2j+2l+1}\overline{\mathcal{D}}^{\mp}\partial_{\mp}^{-1}\overline{\mathcal{J}}_{2j+1}$ appear in eqs. (47) and seem to destroy the chirality properties we are asking for. However, due to the presence of the important projector $(\mathcal{P}_{2k,2k+1} - 1)$ in the operator $P_{2k,2k+1}$ (43), these structures enter eqs. (47) only in the particular combination occuring on the left hand sides of eqs. (48) and thus disappear. So, one might say that $\mathcal{F}^{(l;2p)\mp}_{2j}$ and $\mathcal{F}^{(l;2p)\mp}_{2j+1}$ owe their chirality properties (45) to the projector $(\mathcal{P}_{2k,2k+1} - 1)$ in eq. (43) and the identities (48).

One can straightforwardly verify that the eqs. (47) coincide with the equations that can be derived directly from the equations (46) by applying to them the operator $P_{2l,2l+1}$ and by using the intertwining relations (38). Therefore, eqs. (46) and (47) are consistent with the intertwining relations (38). Comparison of eqs. (47) and eqs. (46) shows that the operator $P_{2l,2l+1}$ preserves the chirality of the first $2l$ terms in the products of eqs. (46) but flips the chirality of the remaining $p-2l$ factors. Moreover, one can easily see from its definition (43) that $P_{2l,2l+1}$ always preserves the chirality of the first term in a product. Therefore, the chirality properties of the functionals $\{\mathcal{F}^{(l;2p)\mp}_{2j}, \mathcal{F}^{(l;2p)\mp}_{2j+1}\}$ are identical to those of $\{\mathcal{F}^{(2p)\mp}_{2j}, \mathcal{F}^{(2p)\mp}_{2j+1}\}$. Furthermore, it is obvious by induction that the same arguments can be applied to each functional from the set (44) since they are recursively related by the operator $P_{2l,2l+1}$. Thus, we are led to the conclusion that the functionals $\mathcal{F}^{m\dots l(2p)\mp}_{j}$ (44) in fact satisfy the chirality constraints (45), which in turn implies that $\alpha^{(2p)\pm}_{0,j} - \alpha^{(2p)\pm}_{0,j-1}$ satisfy the constraints (36). This concludes our proof (of half the conjecture).

5 Further compelling evidence

Unfortunately, we are unable to prove the remaining constraints (37) for $\alpha_{0,j}^{(2p)\pm} - \alpha_{0,j-1}^{(2p)\pm}$ and establish our *conjecture* beyond any doubt. Short of that, we have explicitly verified the *conjecture* for the first three flows $\frac{\partial}{\partial t_l^+}$ resulting from the equations (33). After rather tedious calculations, the flows can be represented in the following form:

$$\frac{\partial}{\partial t_1^+}\mathcal{J} = \partial_+\mathcal{J}, \qquad \frac{\partial}{\partial t_1^+}\overline{\mathcal{J}} = \partial_+\overline{\mathcal{J}}, \tag{49}$$

$$\begin{aligned} \frac{\partial}{\partial t_2^+}\mathcal{J} &= -\partial_+^2\mathcal{J} - \mathcal{D}_+\mathcal{D}_-\left[2\partial_+(\mathcal{J}\partial_-^{-1}\overline{\mathcal{J}}) - (\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J})^2\right], \\ \frac{\partial}{\partial t_2^+}\overline{\mathcal{J}} &= +\partial_+^2\overline{\mathcal{J}} - \overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\left[2\partial_+(\overline{\mathcal{J}}\partial_-^{-1}\mathcal{J}) - (\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}})^2\right], \end{aligned} \tag{50}$$

$$\begin{aligned} \frac{\partial}{\partial t_3^+}\mathcal{J} &= \partial_+^3\mathcal{J} + \mathcal{D}_+\mathcal{D}_-\Big\{3\partial_+\Big[(\partial_+\mathcal{J})\partial_-^{-1}\overline{\mathcal{J}} + (\mathcal{J}\partial_-^{-1}\overline{\mathcal{J}})\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}} \\ &\quad -\frac{1}{2}(\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J})^2\Big] - (\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J})^3 - 3(\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J})^2\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}} \\ &\quad +3\mathcal{J}(\partial_-^{-1}\partial_+\overline{\mathcal{J}})\,\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J} + 3\mathcal{J}\partial_-^{-1}\partial_+(\overline{\mathcal{J}}\,\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J}) \\ &\quad +3(\overline{\mathcal{D}}^+\mathcal{J})(\overline{\mathcal{D}}^-\partial_-^{-1}\partial_+\mathcal{J})\partial_-^{-1}\overline{\mathcal{J}} - 3(\overline{\mathcal{D}}^+\mathcal{J})\overline{\mathcal{D}}^-\partial_-^{-1}\partial_+(\mathcal{J}\partial_-^{-1}\overline{\mathcal{J}})\Big\}, \\ \frac{\partial}{\partial t_3^+}\overline{\mathcal{J}} &= \partial_+^3\overline{\mathcal{J}} + \overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\Big\{3\partial_+\Big[-(\partial_+\overline{\mathcal{J}})\partial_-^{-1}\mathcal{J} + (\overline{\mathcal{J}}\partial_-^{-1}\mathcal{J})\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J} \\ &\quad +\frac{1}{2}(\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}})^2\Big] - (\mathcal{D}_+\mathcal{D}_-\partial^{-1}\overline{\mathcal{J}})^3 - 3(\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}})^2\overline{\mathcal{D}}^+\overline{\mathcal{D}}^-\partial_-^{-1}\mathcal{J} \\ &\quad +3\overline{\mathcal{J}}(\partial_-^{-1}\partial_+\mathcal{J})\,\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}} + 3\overline{\mathcal{J}}\partial_-^{-1}\partial_+(\mathcal{J}\mathcal{D}_+\mathcal{D}_-\partial_-^{-1}\overline{\mathcal{J}}) \\ &\quad +3(\mathcal{D}_+\overline{\mathcal{J}})(\mathcal{D}_-\partial_-^{-1}\partial_+\overline{\mathcal{J}})\partial_-^{-1}\mathcal{J} - 3(\mathcal{D}_+\overline{\mathcal{J}})\mathcal{D}_-\partial_-^{-1}\partial_+(\overline{\mathcal{J}}\partial_-^{-1}\mathcal{J})\Big\}, \end{aligned} \tag{51}$$

where $\mathcal{J} \equiv \mathcal{J}_{2j}(x^+,\theta^+,\eta^+;x^-,\theta^-,\eta^-)$ and $\overline{\mathcal{J}} \equiv \overline{\mathcal{J}}_{2j}(x^+,\theta^+,\eta^+;x^-,\theta^-,\eta^-)$. The right hand sides are obviously consistent with the chirality constraints (34).

Considering the proved first half (i.e. eqs. (36)) of the conjectured chirality constraints (36)–(37) and the proof of the *conjecture* for the one-dimensional reduction to the $N = 4$ supersymmetric Toda chain hierarchy [10], it is reasonable to believe that our *conjecture* is valid for the whole hierarchy of flows $\frac{\partial}{\partial t_l^\pm}$.

6 Five real forms

Direct verification shows that the flows (49)–(51) admit the following five inequivalent[5] complex conjugations:

$$(\mathcal{J},\overline{\mathcal{J}})^* = -(\mathcal{J},\overline{\mathcal{J}}), \qquad (x^\pm,\theta^\pm,\eta^\pm)^* = (-x^\pm,\theta^\pm,-\eta^\pm), \qquad (t_l^\pm)^* = (-1)^l t_l^\pm, \tag{52}$$

[5] We mean that it is not possible to relate them via obvious symmetries, perhaps, some elusive equivalence exists, nevertheless, cf. [11].

$$
\begin{aligned}
&(\mathcal{J},\overline{\mathcal{J}})^{\bullet} = (\ \mathcal{J} - \mathcal{D}_{-}\mathcal{D}_{+}\ln\overline{\mathcal{J}},\ \overline{\mathcal{J}}),\\
&(x^{\pm},\theta^{\pm},\eta^{\pm})^{\bullet} = (-x^{\pm},\theta^{\pm},-\eta^{\pm}), \qquad (t_l^{\pm})^{\bullet} = -t_l^{\pm},
\end{aligned}
\tag{53}
$$

$$
(\mathcal{J},\overline{\mathcal{J}})^{\star} = (\overline{\mathcal{J}},\ \mathcal{J}), \qquad (x^{\pm},\theta^{\pm},\eta^{\pm})^{\star} = (-x^{\pm},\theta^{\pm},\eta^{\pm}), \qquad (t_l^{\pm})^{\star} = -t_l^{\pm}, \tag{54}
$$

$$
(\mathcal{J},\overline{\mathcal{J}})^{\dagger} = -(\mathcal{J},\overline{\mathcal{J}}), \qquad (x^{\pm},\theta^{\pm},\eta^{\pm})^{\dagger} = (-x^{\pm},i\eta^{\pm},i\theta^{\pm}), \qquad (t_l^{\pm})^{\dagger} = (-1)^l t_l^{\pm}, \tag{55}
$$

$$
(\mathcal{J},\overline{\mathcal{J}})^{\ddagger} = (\mathcal{J},\overline{\mathcal{J}}), \qquad (x^{\pm},\theta^{\pm},\eta^{\pm})^{\ddagger} = (-x^{\mp},\theta^{\mp},-\eta^{\mp}), \qquad (t_l^{\pm})^{\ddagger} = (-1)^l t_l^{\mp}. \tag{56}
$$

These involutions extract five inequivalent real forms of the hierarchy. In particular, the flows of the real form corresponding to the conjugation (54) reproduce the algebra of *real* $N=(2|2)$ supersymmetry. We use the standard convention regarding complex conjugation of products involving odd operators and functions (see, e.g., the books [12]). In particular, if $\mathbb{D}$ is some even differential operator acting on a superfield F, we define the complex conjugate of $\mathbb{D}$ by $(\mathbb{D}F)^* = \mathbb{D}^* F^*$.

A combination of the two complex conjugations (54) and (53), when applied twice, generates a manifestly $N=(2|2)$ supersymmetric form of the $N=(2|2)$ Toda lattice equations, (32):

$$
\mathcal{J}^{\star\bullet\star\bullet} = \mathcal{J} - \mathcal{D}_{-}\mathcal{D}_{+}\ln\overline{\mathcal{J}}, \qquad \overline{\mathcal{J}}^{\star\bullet\star\bullet} = \overline{\mathcal{J}} - \overline{\mathcal{D}}^{-}\overline{\mathcal{D}}^{+}\ln\mathcal{J}^{\star\bullet\star\bullet}. \tag{57}
$$

In other words, if the set $\{\mathcal{J},\overline{\mathcal{J}}\}$ is a solution of the $N=(2|2)$ Toda lattice hierarchy, then the set $\{\mathcal{J}^{\star\bullet\star\bullet},\overline{\mathcal{J}}^{\star\bullet\star\bullet}\}$, related to the former by eqs. (57), is also a solution of the hierarchy.

Finally, a combination of the two complex conjugations (52) and (53) generates a second-order discrete symmetry of the flows $\frac{\partial}{\partial t_{2l+1}^{\pm}}$,

$$
(\mathcal{J},\overline{\mathcal{J}})^{\bullet *} = -(\mathcal{J} + \mathcal{D}_{-}\mathcal{D}_{+}\ln\overline{\mathcal{J}},\overline{\mathcal{J}}), \qquad (\mathcal{J},\overline{\mathcal{J}})^{\bullet *\bullet *} = (\mathcal{J},\overline{\mathcal{J}}). \tag{58}
$$

7 Conclusion

In this letter we have proposed an $N=(2|2)$ superfield formulation of the $N=(2|2)$ supersymmetric Toda lattice hierarchy and have constructed five different real forms in $N=(2|2)$ superspace.

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